Topologically Faithful Implicit Curve Approximation by Constrained Least Squares

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Introduction

Keren [2004] has proposed a noteworthy implicit curve approximation method which **guarantees the topology** of the estimated curve, but it involves solving a high-degree non-linear optimization problem.

We propose a topologically faithful method whose optimization is a quadratic programming problem.

Method by Keren [2004]

The idea is to construct a homeomorphism from the data points to the unit circle by using the following theorems.

- Let f_1, f_2, g_1, g_2 be strictly monotonic increasing functions, then
  $$(x, y) \to (f_1 - f_2, g_1 + g_2)$$
  is a homeomorphism.
- Any one variable positive polynomial (=the derivative of increasing polynomial) is sum of squares.

If homeomorphism h is approximated, the implicit curve of $||h|| - 1$ is the estimated curve.

Mapping to a Polygon

Keren [2004] uses the distance from the unit circle, which introduces non-linearity. We propose to use a **polygon** as a target. Because we use a homeomorphism, it is sufficient to fit each point to the linear function of the corresponding edge. Thus, the objective function for a point assigned to the edge $ax + by = c$ can be formulated as the **linear least squares**:

$$(ah_1(x_i, y_i) + bh_2(x_i, y_i) - c)^2.$$

This method requires an ordering of the data points, and points should be divided into each edge. Any kind of polygon can be used as a target, so we can exploit our prior knowledge on the shape.

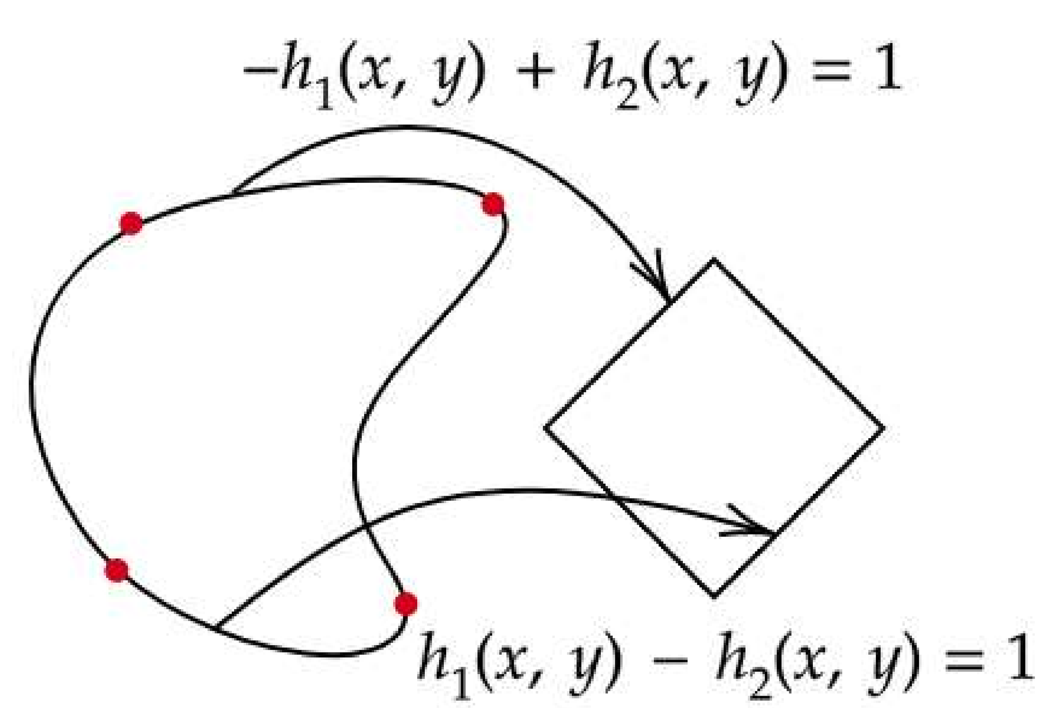
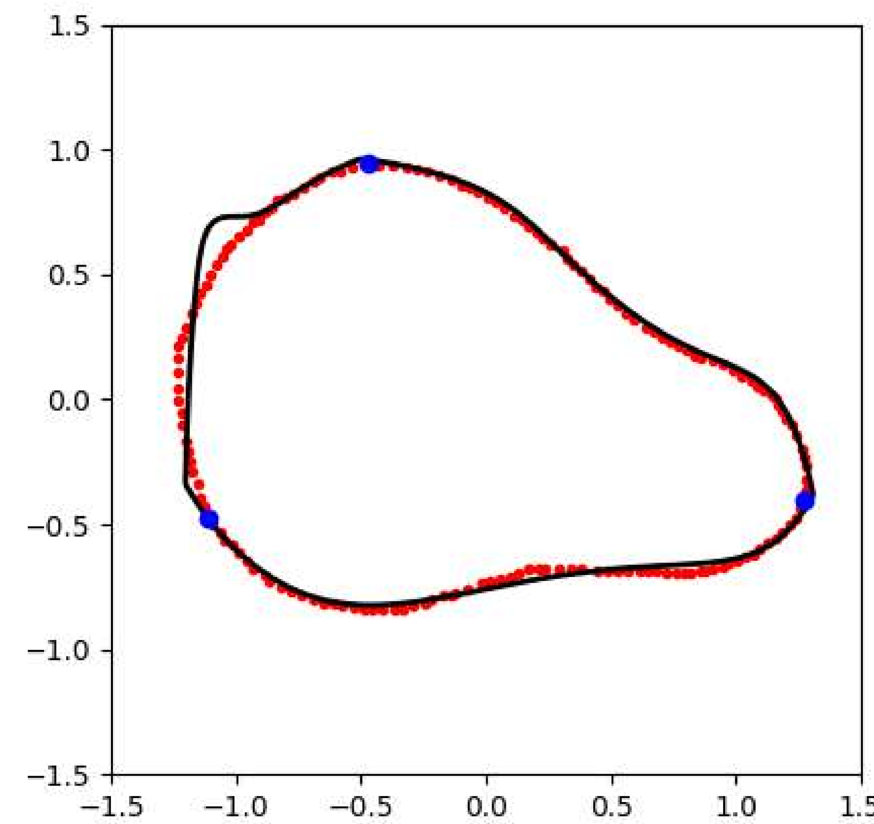
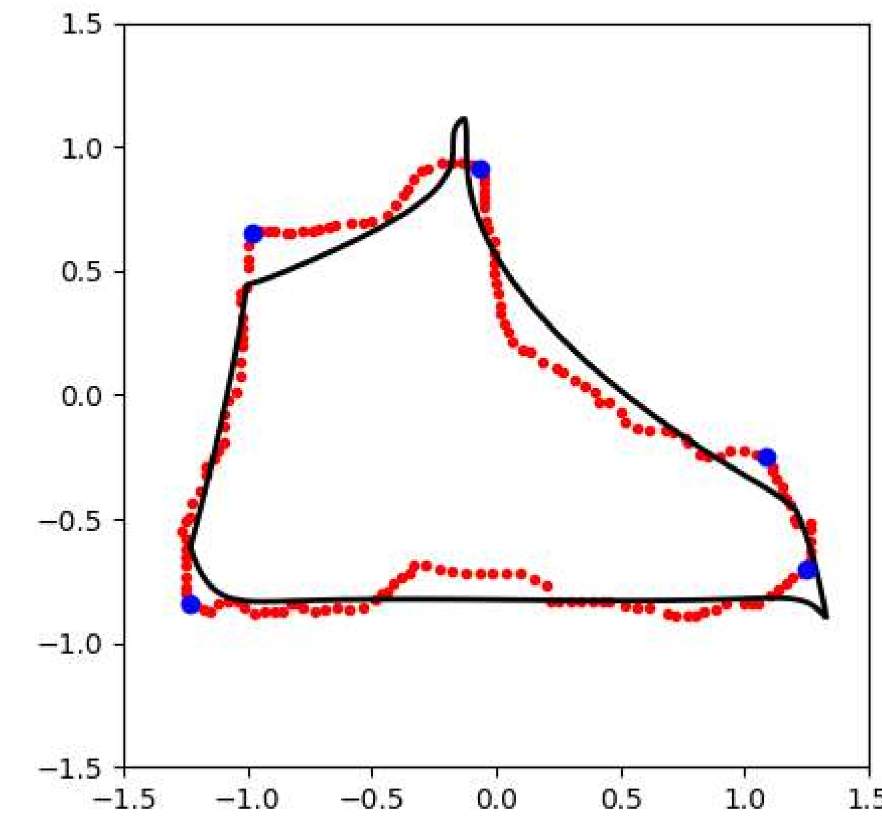


Figure: The curve is divided into four edges.



(a) Implicit curve trained as the mapping to a triangle.



(b) The representation power is not sufficient for complex shape.

- Data points (red).
- estimated curve (black).
- data points corresponding to vertices of a polygon (blue).

Positive Polynomial

The method by Keren [2004] also introduces non-linearity when it constructs a positive polynomial as sum of squares explicitly. In order to avoid this non-linearity, we propose to use the **linear sufficient condition** proposed by Ghasemi and Marshall [2010].

d-degree (d: even) polynomial $\Sigma\, c_i x^i$ is positive if

$$c_0 \geq \Sigma_{i:\text{odd}}|c_i| - \Sigma_{i:\text{even}} \min\{0, |c_i|\}$$
$$c_d \geq \Sigma_{i:\text{odd}} i|c_i|/d - \Sigma_{i:\text{even}} \min\{0, |c_i|/d\}.$$

By using the mapping to a polygon and the linear sufficient condition, we can formulate the problem as a **convex quadratic program**.

Conclusion

We constructed a topologically faithful implicit curve approximation that can be solved using convex quadratic programming, which is much easier than the high-degree non-linear optimization required in prior work.

References

Mehdi Ghasemi and Murray Marshall. Lower bounds for a polynomial in terms of its coefficients. *Arch. Math*, 95:343–353, 2010. doi: 10.1007/s00013-010-0179-0.

Daniel Keren. Topologically Faithful Fitting of Simple Closed Curves. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 26(1):118–123, 2004. ISSN 01628828. doi: 10.1109/TPAMI.2004.1261095.